Sociology 376 **Exam 1** **Spring 2012** **Prof Montgomery**

Answer all questions. 210 points possible. Explanations can be brief. You may be time-constrained, so please allocate your time carefully.

[HINT: For several questions on this exam, it may be useful to know that

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \text{ implies } A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.]$$

1) [50 points] Suppose individuals can migrate between country 1 and country 2. Further suppose that the transition matrix for 2008 was

$$\begin{bmatrix} .96 & .04 \\ .15 & .85 \end{bmatrix}$$

and that the transition matrix was identical for 2009. (These transition matrices are specified so that element (i,j) indicates the probability of transition from country i to country j over the calendar year.) Over the course of the two years, the transition matrix was

$$\begin{bmatrix} .956 & .044 \\ .165 & .835 \end{bmatrix}$$

(This transition matrix is specified so that element (i,j) indicates the probability of transition from country i to country j over the period from January 1, 2008 to December 31, 2009.)

a) What are the key assumptions maintained in any Markov chain process? Based on the information above, does migration between these countries seem to follow a Markov chain process? Explain why (or why not), showing any relevant calculations.

b) One might hypothesize that some residents of countries 1 and 2 are "stayers" who never migrate, while others are "movers" whose migration is described by a Markov chain process. What equation describes the dynamics for the "mover-stayer" model?

c) Suppose that 50% of the initial residents (on January 1, 2008) of both countries were stayers (while the others were movers). Given the observed (one-year) transition matrix for the overall population, compute the implied (one-year) transition matrix for movers.

d) Given the assumption and computation in part (c), what is the predicted two-year transition matrix? Comparing this prediction to the observed (two-year) transition matrix, does the mover-stayer model appear more or less accurate than the simple Markov chain model (which assumed no stayers)? To generate a better prediction, how would you further adjust the assumed proportion of stayers in each country?

2) [30 points] Consider a society where individuals belong to religion 1 or 2 or 3 or 4. Intergenerational transitions between religions are governed by the transition matrix

```
P =
    0.5000    0.3000    0.2000         0
    0.2500    0.6000    0.1000    0.0500
    0.0500    0.1000    0.6500    0.2000
         0    0.1000    0.3000    0.6000
```

where P(i,j) indicates the probability that a parent with religion i has a child with religion j. The reproduction matrix is given by

```
R =
    1.0000         0         0         0
         0    1.5000         0         0
         0         0    2.0000         0
         0         0         0    2.5000
```

where R(i,i) indicates the average number of children born to a parent with religion i.

Use the Matlab computations below to answer the following questions.

a) Normalizing the size of the initial (generation 0) population to 1, and assuming that everyone initially belongs to religion 2, what is the (frequency) distribution of the population over the next 3 generations?

b) Find the long-run growth factor and the limiting distribution (as a probability vector). Explain how these can be determined from the eigenvalues and eigenvectors of the (RP)' matrix.

c) Why is it appropriate to compute the eigenvalues and eigenvectors of (RP)' instead of RP?

```
>> R*P

ans =
    0.5000    0.3000    0.2000         0
    0.3750    0.9000    0.1500    0.0750
    0.1000    0.2000    1.3000    0.4000
         0    0.2500    0.7500    1.5000

>> (R*P)^2

ans =
    0.3825    0.4600    0.4050    0.1025
    0.5400    0.9712    0.4613    0.2400
    0.2550    0.5700    2.0400    1.1350
    0.1688    0.7500    2.1375    2.5688

>> (R*P)^3

ans =
    0.4043    0.6354    0.7489    0.3503
    0.6803    1.1884    1.0333    0.6173
    0.5453    1.2813    3.6398    2.5613
    0.5794    1.7953    4.8516    4.7644
```

```
>> [eigvec, eigval] = eig((R*P)')

eigvec =

    0.1216    0.8854    0.5253    0.0926
    0.2981   -0.4202    0.7206    0.3001
    0.7281   -0.1789    0.2644   -0.8337
    0.6052    0.0860   -0.3672    0.4541


eigval =

    2.0182         0         0         0
         0    0.3018         0         0
         0         0    1.0647         0
         0         0         0    0.8153
```

3) [60 points] At Alpha University, faculty are hired as assistant professors. At the end of each year, an assistant professor can be promoted to associate professor (with probability x) or remain an assistant professor (with probability y) or be fired (with probability 1-x-y). Fired faculty members are never reappointed (i.e., they remain fired forever). At the end of each year, an associate professor can be promoted to full professor (with probability z) or remain an associate professor (with probability 1-z). Full professors are reappointed as full professors every year (i.e., they remain full professors forever).

a) What are the states of this Markov chain? Is it an absorbing chain? If so, which states are absorbing?

b) Draw the transition diagram for this process. Then specify the transition matrix in canonical form. [HINT: The matrix should take the form P = $\begin{bmatrix} I & 0 \\ R & Q \end{bmatrix}$ with I, 0, R, and Q submatrices.]

c) In general, given a transition matrix in canonical form, what is the equation for the fundamental matrix N? Find the fundamental matrix N for this problem. [HINT: The elements of N will depend on the parameters x, y, and z.]

d) For someone just appointed as an assistant professor, what is the expected number of years she'll be an assistant professor? What is the expected number of years she'll be an associate professor?

e) For someone just appointed as an associate professor, what is the expected number of years she'll be an associate professor?

f) For some just appointed as an assistant professor, what is the probability that she'll ultimately be fired? What is the probability that she'll eventually become a full professor?

[HINT: P = $\begin{bmatrix} I & 0 \\ R & Q \end{bmatrix}$ implies P^{∞} = $\begin{bmatrix} I & 0 \\ NR & 0 \end{bmatrix}$.]

4) [70 points] At Beta University, faculty are reviewed every 5 years. At the 5^{th} year review, the faculty member is retained with probability x, and fired with probability 1-x. At the 10^{th} year review, the faculty member is retained with probability y, and fired with probability 1-y. Faculty are always retained at the 15^{th} year and 20^{th} year and 25^{th} year review, and always retire when they've been with the university for exactly 30 years (so there is no 30^{th} year review).

The university hires new faculty only when old faculty retire. More precisely, suppose that the university hires 2 new faculty members upon each retirement.

a) Draw the transition diagram for this process (ignoring the absorbing state of being fired or retired). Then specify the Leslie matrix (using the demography convention so that transitions are from the *columns* to the *rows* of the matrix).

b) Compute the probability that a new faculty member will "survive" from initial hire to each age class. Then compute the expected number of years that the new faculty member will spend at the university.

c) Compute the gross reproduction rate (GRR) and net reproduction rate (NRR) for this example. Under what condition is the size of the faculty increasing or decreasing? If faculty are retained at the 5^{th} year review with probability x = .8, what value of y would maintain a constant faculty size? (You can assume these values of x and y for any computations below.)

d) Is the Leslie matrix for this example irreducible? Is it primitive? Explain these conditions, and show why the matrix does (or doesn't) meet them using the transition diagram in part (a).

e) Given the answer to part (d), will the population of faculty converge to a unique "stable growth" equilibrium (with a constant growth rate, and a constant proportion of faculty in each age class) for any initial condition? If so, what is the limiting distribution? If not, explain why the distribution won't converge.

f) The university is contemplating a new policy for hiring faculty. Under this new policy, one new faculty member would be hired when an existing faculty member reaches year 25, and another new faculty member would be hired when an existing faculty member reaches retirement (in year 30). (In this way, the university would still make 2 hires for each retiring faculty member, but one of those hires would be made 5 years earlier.) Answer parts (d) and (e) again given this new hiring policy.

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1a) [10 pts] Markov chain models assume (1) the transition matrix doesn't change over time and (2) there is no history dependence (i.e., transition probabilities depend only on the current state). Here, assumption (1) seems to be met, but assumption (2) is not. If there was no history dependence, then the product of the one-year transition matrices should equal the 2-year transition matrix. But matrix multiplication reveals

$$\begin{bmatrix} .96 & .04 \\ .15 & .85 \end{bmatrix} \begin{bmatrix} .96 & .04 \\ .15 & .85 \end{bmatrix} = \begin{bmatrix} .9276 & .0724 \\ .2715 & .7285 \end{bmatrix} \neq \begin{bmatrix} .956 & .044 \\ .165 & .835 \end{bmatrix}$$

b) [10 pts] Letting S denote the stayer matrix (giving the initial proportion of individuals in each class who are stayers) and letting M denote the transition matrix for movers, the observed transition matrix for period t is given by $A_t = S + (I\text{-}S)M^t$. Given the initial condition x_0 (row vector), the population distribution in period t would be $x_t = x_0\,[S + (I\text{-}S)M^t]$.

c) [20 pts] $A_1 = S + (I\text{-}S)M$ implies $M = (I–S)^{-1}(A_1 – S)$

Here, $I–S = \begin{bmatrix} .5 & 0 \\ 0 & .5 \end{bmatrix}$ which (using the hint) implies $(I–S)^{-1} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$

and thus $M = (I–S)^{-1}(A_1 – S) = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} .46 & .04 \\ .15 & .35 \end{bmatrix} = \begin{bmatrix} .92 & .08 \\ .3 & .7 \end{bmatrix}$

d) [10 pts] Given S and M, the predicted two-year matrix is

$$A_2 = S + (I–S)M^2 = \begin{bmatrix} .5 & 0 \\ 0 & .5 \end{bmatrix} + \begin{bmatrix} .5 & 0 \\ 0 & .5 \end{bmatrix} \begin{bmatrix} .92 & .08 \\ .3 & .7 \end{bmatrix} \begin{bmatrix} .92 & .08 \\ .3 & .7 \end{bmatrix} = \begin{bmatrix} .9352 & .0648 \\ .243 & .757 \end{bmatrix}$$

Note that this predicted matrix doesn't precisely match the observed two-year matrix, but is closer than A_1A_1 because there is more "weight" on the main diagonal. You can fit the mover-stayer model perfectly to the observed matrix by further increasing the proportions of stayers. [Indeed, I created the example by assuming $S = \begin{bmatrix} .9 & 0 \\ 0 & .7 \end{bmatrix}$ and $M = \begin{bmatrix} .6 & .4 \\ .5 & .5 \end{bmatrix}$.]

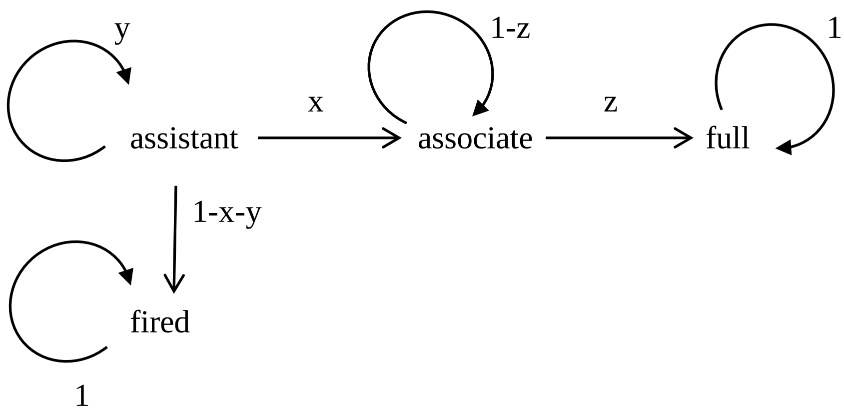
2a) [10 pts] The answer is given by the second row of the RP and $(RP)^2$ and $(RP)^3$ matrices: [.375 .9 .15 .075] and [.54 .9712 .4613 .2400] and [.6803 1.1884 1.0333 .6173].

b) [12 pts] The long-run growth factor is the dominant eigenvalue of the Leslie matrix (that is, the eigenvalue with the largest absolute value) and the limiting distribution is the associate eigenvector (normalized as a probability vector). Here, the growth factor is 2.0182 and the limiting distribution is [.0694 .1701 .4153 .3452].

c) [8 pts] The eigenvalues and eigenvectors of the matrix A are determined by the equation $\lambda x = Ax$ where λ is the (scalar) eigenvalue, and the eigenvector x is specified as a column vector. The equation for the long-run outcome is $x\lambda = xRP$ where x is specified as a row vector. We need to transpose both sides of this equation to obtain the eigenvector equation $\lambda x' = (RP)'x'$.

3a) [8 pts] The states are (1) fired, (2) full professor, (3) assistant professor, (4) associate professor. It is an absorbing chain (because it has absorbing states, and every state can reach an absorbing state). States (1) fired and (2) full professor are absorbing. [Note that I've listed the absorbing states first, consistent with the transition matrix for part (b).]

b) [12 pts] The transition diagram is



and the transition matrix is $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1-x-y & 0 & y & x \\ 0 & z & 0 & 1-z \end{bmatrix}$

c) [15 pts] The fundamental matrix N = I + Q + Q^2 + … = $(I–Q)^{-1}$

For the present problem, I – Q = $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} y & x \\ 0 & 1-z \end{bmatrix} = \begin{bmatrix} 1-y & -x \\ 0 & z \end{bmatrix}$

and hence (using the hint on the first page of the exam) we obtain

$$N = (I–Q)^{-1} = \frac{1}{(1-y)z}\begin{bmatrix} z & x \\ 0 & 1-y \end{bmatrix} = \begin{bmatrix} 1/(1-y) & x/[(1-y)z] \\ 0 & 1/z \end{bmatrix}$$

d) [10 pts] The first row/column of the Q matrix corresponds to the assistant professor state; the second row/column corresponds to the associate professor state. For someone currently an assistant professor, the expected number of years as an assistant professor = N(1,1) = 1/(1-y) and the expected number of years as an associate professor = N(1,2) = x/[(1-y)z].

e) [5 pts] For someone just appointed to associate professor, the expected number of years as an associate professor = N(2,2) = 1/z.

f) [10 pts] The NR matrix gives the probability that, starting from each non-absorbing state, the process is absorbed in each absorbing state. Here, NR = $\begin{bmatrix} (1-x-y)/(1-y) & x/(1-y) \\ 0 & 1 \end{bmatrix}$.
Thus, a new assistant professor is ultimately fired with probability (1-x-y)/(1-y) and eventually becomes a full professor with probability x/(1-y).

4a) [14 pts] The transition diagram is

$$\begin{tikzcd} 0 \arrow[r, "x"] & 5 \arrow[r, "y"] & 10 \arrow[r, "1"] & 15 \arrow[r, "1"] & 20 \arrow[r, "1"] & 25 \arrow[lllll, bend right, "2"'] \end{tikzcd}$$

and the Leslie matrix is $\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 2 \\ x & 0 & 0 & 0 & 0 & 0 \\ 0 & y & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$

b) [12 pts] The faculty member survives to the 5th review with probability x, and to the 10th and 15th and 20th and 25th with probability xy. The expected time in the university is thus (5 years)*(1+x+xy+xy+xy+xy) = 5(1+x+4xy) years.

c) [12 pts] GRR = 2. NRR = 2xy. The population is growing when NRR > 1 (and is shrinking when NRR < 1). If x = .8, the university would need to set y = .625 to maintain constant faculty size.

d) [12 pts] For this example, the Leslie matrix is irreducible but is not primitive. The matrix is irreducible because every state can reach every other state (which is apparent from the transition diagram). The matrix is not primitive. The transition diagram in part (a) shows only one cycle, which is of length 6. Because there are not two cycles whose lengths are relatively prime, the matrix is cyclical rather than primitive. (You can also establish that the Leslie matrix is not primitive by raising it to higher powers and observing that the zeros never "fill in." But I asked you to establish the result using the transition diagram directly.)

e) [10 pts] No, the Leslie matrix is not primitive (nor is it centered). Thus, the distribution of faculty will not converge to a unique limiting distribution for any initial condition. Instead, the distribution will follow a 6-period cycle. For instance, given the initial condition [1 0 0 0 0 0], the pattern would be [1 0 0 0 0 0] then [0 .8 0 0 0 0] then [0 0 .5 0 0 0] then [0 0 0 .5 0 0] then [0 0 0 0 .5 0] then [0 0 0 0 0 .5] then [1 0 0 0 0 0], which was the initial condition.

f) [10 pts] The Leslie matrix is now irreducible and primitive. The transition diagram would now indicate one cycle of length 5 and another of length 6. Because there are two cycles whose lengths are relatively prime, the matrix is primitive. Consequently, the distribution of faculty will converge to a unique limiting distribution for any initial condition.

Sociology 376 Exam 2 Spring 2012 Prof Montgomery

Answer all questions. 210 points possible.

1) [100 pts] A restaurant in Los Angeles has two types of customers: celebrities and fans. Fans are drawn to the restaurant by celebrities, while celebrities prefer to go to the restaurant when the number of fans is not too small (they need the publicity) and not too large (they become overwhelmed by the attention). Both fans and celebrities have adaptive expectations. Formally, let F_t denote the number of fans at restaurant in period t, and let C_t denote the number of celebrities at restaurant in period t. Assume that the dynamics are given by

$$F_{t+1} = \max\{-200 + 50\, C_t, 0\}$$
$$C_{t+1} = \max\{-10 + (1/5)\, F_t - (1/5000)\, (F_t)^2, 0\}$$

For simplicity, assume that F and C are continuous (non-negative) variables. Note that the restaurant has no capacity constraint (i.e., it has an unlimited number of seats).

a) Derive the equation for the F-nullcline and the C-nullcline, then plot these curves in F-C space. [HINT: Assume that both F_{t+1} and C_{t+1} are positive to obtain the nullclines. Your graph doesn't need to be perfect, but should be qualitatively correct and well labeled. You should also draw your graph large because you'll use it again in parts (b) and (e) and (f).]

b) Complete the phase diagram you started in part (a) by adding arrows to denote the direction of ΔF and ΔC in each region of the diagram.

c) Solve numerically for the two interior (non-zero) equilibria of this model. [HINT: Recall that the quadratic equation $ax^2 + bx + c = 0$ has solutions $x = [-b \pm \text{sqrt}(b^2 - 4ac)] / 2a$].

d) Assess the stability of each interior equilibrium by computing the Jacobian matrix, evaluating it at each equilibrium, and finding the eigenvalues.

[HINT: The matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ has eigenvalues $\lambda = (1/2)\,[a + d \pm \text{sqrt}(a^2 + 4bc - 2ad + d^2)]$.]

e) Suppose $F_0 = 900$ and $C_0 = 8$. Compute the trajectory for the next 5 periods. What's the long-run behavior of the system? Illustrate this trajectory on your phase diagram. Intuitively, how would the dynamics differ if the model was respecified in continuous time?

f) The owner of the restaurant in considering whether to cap the number of fans at 500 per period. What equilibrium would this create? Would this equilibrium be stable? Briefly explain, using your phase diagram.

2) [80 pts] Under the replicator dynamics, the distribution of strategies evolves according to the matrix equation

$$\frac{dx}{dt} = \text{diag}(x)\,(Ax - x'Ax)$$

where x is an (n × 1) column vector giving the distribution of strategies, and A is the (n × n) payoff matrix for a symmetric game (with payoffs specified from the row player's perspective).

a) Briefly discuss the specification of the replicator equation. In particular, what is the interpretation of Ax? What is the interpretation of x'Ax? Why is it reasonable for the sign of dx(i)/dt to depend on the sign of [(Ax)(i) – x'Ax]?

b) Suppose A = $\begin{bmatrix} 0 & 1 \\ 1/2 & 0 \end{bmatrix}$ and x = $\begin{bmatrix} p \\ 1-p \end{bmatrix}$. Find the equation for dp/dt. What are the equilibrium values of p? Which equilibria are stable? Which are unstable? Briefly explain, plotting dp/dt (as a function of p) to illustrate your answer.

c) Suppose a third strategy is added to the game so that A becomes $\begin{bmatrix} 0 & 1 & 1 \\ 1/2 & 0 & 1/2 \\ \beta & \beta & 0 \end{bmatrix}$.

Further suppose the population is initially in the interior equilibrium from part (b) (with population share p* playing strategy 1, population share 1-p* playing strategy 2, and no one playing strategy 3). Verify that this outcome is an equilibrium of the replicator dynamics (showing the relevant computations). Under what condition (on β) is this equilibrium unstable? [HINT: When can strategy 3 "invade" the initial population?] Briefly explain (again showing the relevant computation).

d) Given β ≤ 1 and the condition on β found in part (c), it can be shown there is a unique interior equilibrium (all three strategies are "active") that is globally stable (all points in the interior of the simplex are in the basin of attraction). Find this equilibrium. [HINT: Under the replicator dynamics, all active strategies must generate the same expected payoff in equilibrium.]

e) What is the interpretation of the A matrix in the paper by Montgomery (*American Economic Journal* 2011) on intergenerational cultural transmission? In particular, how would you interpret the matrix in part (c)? Explain how the survival/success of trait 3 depends on the parameter β.

3) [30 points]
a) Why are two-sex population models necessarily non-linear? What types of outcomes become possible in two-sex models that were not possible in one-sex models?

b) What specification of the matching matrix was adopted in the paper by Montgomery (*American Journal of Sociology* 2012) on racial dynamics? Would this specification remain feasible if the racial distributions of males and females differed? Explain.

Sociology 376 Exam 2 Spring 2012 Solutions

1a) [25 pts] If we assume that F_{t+1} and C_{t+1} are positive, the dynamics become

$$F_{t+1} = -200 + 50\ C_t$$
$$C_{t+1} = -10 + (1/5)\ Ft - (1/5000)\ F_t^2$$

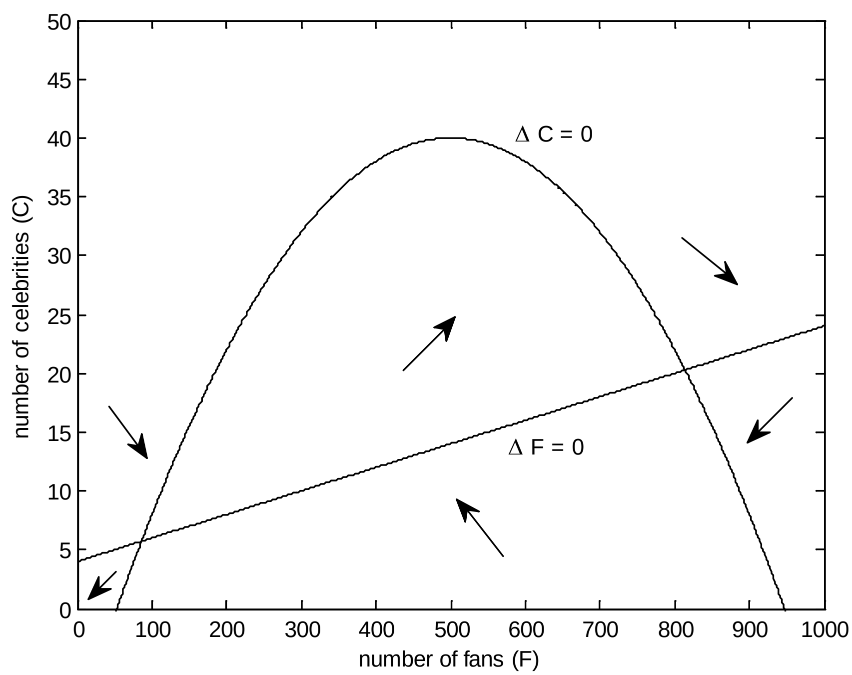
and can be rewritten in "delta" notation as

$$\Delta F = -200 + 50\ C - F$$
$$\Delta C = -10 + (1/5)\ F - (1/5000)\ F^2 - C$$

To find each nullcline, we set ΔF and ΔC to zero:

$$\Delta F = 0 \rightarrow F = -200 + 50\ C \rightarrow C = F/50 + 4$$
$$\Delta C = 0 \rightarrow C = -10 + (1/5)\ F - (1/5000)\ F^2$$

Plotting the nullclines in F-C space:



b) [10 pts] The arrows (shown on the phase diagram above) are determined by

$$\Delta F \geq 0 \rightarrow F \leq -200 + 50C$$
$$\Delta C \geq 0 \rightarrow C \leq -10 + (1/5)\ F - (1/5000)\ F^2$$

1c) [15 pts] The interior equilibria are determined by the intersections of the nullclines:

$$F = -200 + 50\,C$$
$$F = -200 + 50\,[-10 + (1/5)\,F - (1/5000)\,F^2]$$
$$(1/100)\,F^2 - 9\,F + 700 = 0$$
$$F = 450 \pm 364$$

Thus, the equilibria (F^*, C^*) are $(86, 5.72)$ and $(814, 20.28)$

d) [25 pts] The dynamics can be specified as $F_{t+1} = g_1(F_t, C_t)$ and $C_{t+1} = g_2(F_t, C_t)$ where

$$g_1(F,C) = -200 + 50\,C \qquad \text{and} \qquad g_2(F,C) = -10 + (1/5)\,F - (1/5000)\,F^2$$

Thus, the Jacobian matrix is

$$J = \begin{bmatrix} \partial g_1/\partial F & \partial g_1/\partial C \\ \partial g_2/\partial F & \partial g_2/\partial C \end{bmatrix} = \begin{bmatrix} 0 & 50 \\ \frac{1}{5} - \frac{F}{2500} & 0 \end{bmatrix}$$

Alternatively, without using calculus, you could have found the Jacobian matrix by considering small departures from equilibrium, with $F_t = F^* + f_t$ and $C_t = C^* + c_t$ where f_t and c_t are small.

$$F_{t+1} = -200 + 50\,C_t$$
$$F^* + f_{t+1} = -200 + 50\,(C^* + c_t)$$
$$f_{t+1} = 50\,c_t$$

$$C_{t+1} = -10 + (1/5)\,F_t - (1/5000)\,F_t^2$$
$$C^* + c_{t+1} = -10 + (1/5)\,(F^* + f_t) - (1/5000)\,(F^* + f_t)^2$$
$$c_{t+1} = (1/5)\,f_t - (1/5000)\,(2\,F^*\,f_t + f_t^2) \approx (1/5 - F^*/2500)\,f_t$$

and thus $\begin{bmatrix} f_{t+1} \\ c_{t+1} \end{bmatrix} = \begin{bmatrix} 0 & 50 \\ \frac{1}{5} - \frac{F}{2500} & 0 \end{bmatrix} \begin{bmatrix} f_t \\ c_t \end{bmatrix}$

$F^* = 86$ implies $J = \begin{bmatrix} 0 & 50 \\ .1656 & 0 \end{bmatrix}$ and hence $\lambda = \pm\,(1/2)\,\mathrm{sqrt}(4*50*.1656) = \pm\,2.877$

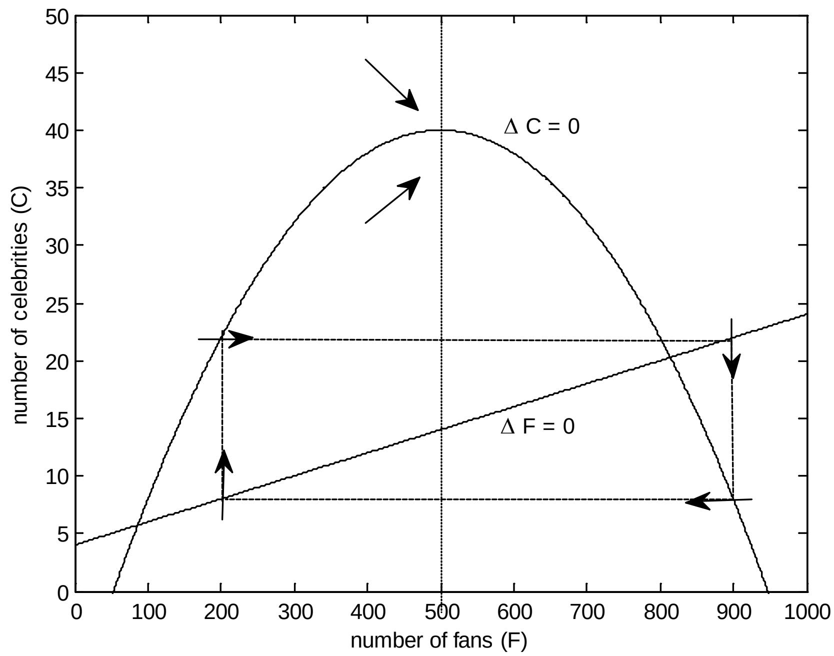
$F^* = 814$ implies $J = \begin{bmatrix} 0 & 50 \\ -.1256 & 0 \end{bmatrix}$ and hence $\lambda = \pm\,(1/2)\,\mathrm{sqrt}(4*50*-.1256) = \pm\,2.51i$

In both cases, the dominant eigenvalue has absolute value greater than 1. Thus, both equilibria are unstable.

e) [15 pts] Using the generator function, we obtain

t	F	C
0	900	8
1	200	8
2	200	22
3	900	22
4	900	8

Thus, the equilibrium is a 4-cycle (i.e., the cycle repeats itself every 4 periods) following the trajectory (dashed lines) on the phase diagram below. Note that each of the 4 outcomes lies on a nullcline.



The continuous time model wouldn't permit these large jumps. Instead, flows would be smooth (as indicated by the arrows in part b), and the trajectory would spiral around the upper equilibrium.

f) [10 pts] This would create an equilibrium at $(F^*, C^*) = (500, 40)$, determined by the intersection of the dotted line and C-nullcline on the phase diagram. As indicated by the arrows, this equilibrium would be stable. Given C in the neighborhood of 40, the number of fans would remain at capacity (F = 500), and the number of celebrities would immediately adjust to C = 40.

2a) [10 pts] Ax is a column vector; element (Ax)(i) is the expected payoff for strategy i. The scalar x'Ax is the average payoff in the population. Intuitively, under the replicator dynamics, x(i) is rising when the expected payoff to strategy i is greater than the average payoff in the population.

b) [25 pts] Substitution into the replicator equation yields

$$\begin{bmatrix} dp/dt \\ d(1-p)/dt \end{bmatrix} = \begin{bmatrix} p & 0 \\ 0 & 1-p \end{bmatrix} \left(\begin{bmatrix} 1-p \\ \left(\frac{1}{2}\right) p \end{bmatrix} - \left(p(1-p) + \left(\tfrac{1}{2}\right) p(1-p)\right) \right)$$

We only need to keep track of one of these equations. Simplifying the first equation, we obtain

$$dp/dt = p\,(1-p)\,[1-(3/2)p]$$

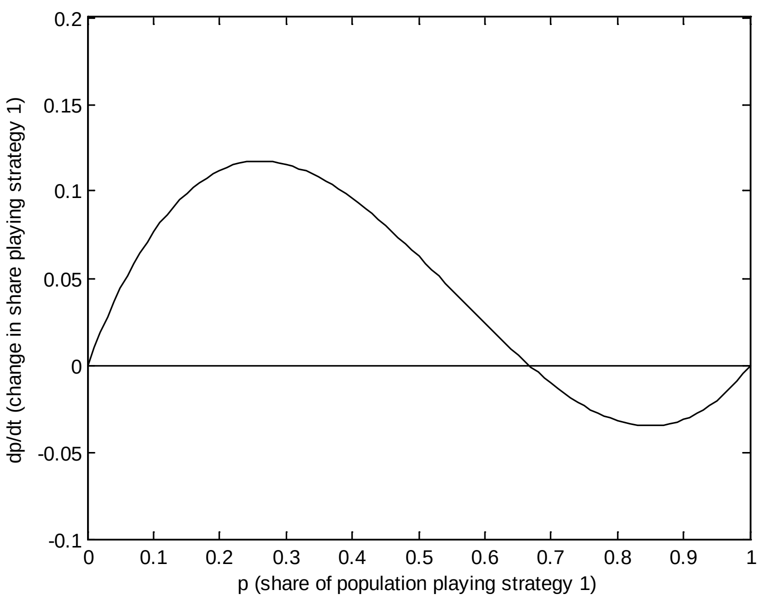
Thus, $dp/dt = 0 \rightarrow p = 0$ or $p = 1$ or $p = 2/3$

Because

$$dp/dt > 0 \text{ for } p \in (0, 2/3)$$
$$dp/dt < 0 \text{ for } p \in (2/3, 1)$$

the interior equilibrium ($p^* = 2/3$) is stable, while the other equilibria are unstable.

Graphically,



2c) [15 pts]

$$x = \begin{bmatrix} 2/3 \\ 1/3 \\ 0 \end{bmatrix} \text{ implies } Ax = \begin{bmatrix} 0 & 1 & 1 \\ 1/2 & 0 & 1/2 \\ \beta & \beta & 0 \end{bmatrix} \begin{bmatrix} 2/3 \\ 1/3 \\ 0 \end{bmatrix} = \begin{bmatrix} 1/3 \\ 1/3 \\ \beta \end{bmatrix} \quad \text{and} \quad x'Ax = 1/3$$

Substitution into the replicator equation yields

$$dx/dt = \begin{bmatrix} 2/3 & 0 & 0 \\ 0 & 1/3 & 0 \\ 0 & 0 & 0 \end{bmatrix} \left(\begin{bmatrix} 1/3 \\ 1/3 \\ \beta \end{bmatrix} - 1/3 \right) = \begin{bmatrix} (\frac{2}{3})(\frac{1}{3} - \frac{1}{3}) \\ (\frac{1}{3})(\frac{1}{3} - \frac{1}{3}) \\ 0(\beta - \frac{1}{3}) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Thus, this outcome is an equilibrium of the replicator dynamics. However, this equilibrium is unstable if strategy 3 can invade the population. This occurs when 3's expected payoff (Ax)(3) = β is greater than the expected payoffs to the active strategies (Ax)(1) = (Ax)(2) = 1/3. Thus, the equilibrium is unstable when β > 1/3.

d) [20 pts] Given $x = \begin{bmatrix} p \\ q \\ 1-p-q \end{bmatrix}$, the expected payoffs are $Ax = \begin{bmatrix} 1-p \\ (\frac{1}{2})(1-q) \\ \beta(p+q) \end{bmatrix}$

In equilibrium, all three strategies earn the same expected payoff: 1–p* = (1/2)(1–q*)= β(p*+q*)

Some (tedious) algebra yields p* = (1+β)/(3β+1), q* = (1–β)/(3β+1), 1-p*-q* = (3β–1)/(3β+1)

e) [10 pts] In that paper, A(i,j) can be interpreted as the "distaste" of trait i for trait j. More precisely, A(i,j) = [V(i,i) – V(i,j)] where V(i,j) is the value that a trait-i parent places on a trait-j child. The matrix in part (c) assumes that trait 1 has the strongest distaste of the other traits, while traits 2 and 3 have weaker distastes for the other traits. (We would need to know if β is greater than or less than ½ to rank traits 2 and 3.) The analysis in parts (c) and (d) demonstrates that trait 3 will not survive in the long run if β < 1/3. If it does survive, its equilibrium share of the population (1–p*–q*) = (3β–1)/(3β+1) is increasing in β. Overall, the model implies that traits with strong distastes for other traits have an evolutionary advantage.

3a) [10 pts] Except in the extreme case of complete homogamy, the matching probabilities in a two-sex model are not constant but depend on the distributions of males and females. This creates the non-linearity in the model. As illustrated in Montgomery (*AJS* 2012), the non-linearity of two-sex models permits multiple equilibria or limit cycles. These outcomes can't arise in one-sex models because their linearity implies a unique equilibrium that is globally stable (i.e., will be reached from any initial condition).

b) [20 pts] In that model, M(i,ij) denotes the probability that a female of race i $\in$ {w,m,b} forms a couple of type ij $\in$ {ww, wm, wb, mw, mm, mb, bw, bm, bb}. The (non-zero) matching probabilities were specified as

$$
\begin{aligned}
M(w,ww) &= 1 - \mu\, x(m) - \beta\, x(b) \\
M(w,wm) &= \mu\, x(m) \\
M(w,wb) &= \beta\, x(b) \\
M(m,mw) &= \mu\, x(w) \\
M(m,mm) &= 1 - \mu\, x(w) - \gamma\, x(b) \\
M(m,mb) &= \gamma\, x(b) \\
M(b,bw) &= \beta\, x(w) \\
M(b,bm) &= \gamma\, x(m) \\
M(b,bb) &= 1 - \beta\, x(w) - \gamma\, x(m)
\end{aligned}
$$

where x(i) denotes the proportion of men of race i, and {μ, β, γ} are parameters reflecting the willingness of individuals to marry partners of other races. Note that $\mu = \beta = \gamma = 0$ implies complete homogamy, and $\mu = \beta = \gamma = 1$ implies random matching.

Any specification of the matching matrix must satisfy two accounting conditions. First, assuming all women marry, each row of the matching matrix to must sum to 1. That is,

$$\sum_j M(i,ij) = 1 \quad \text{for all i}$$

Second, the "demand" and "supply" of men must be in balance. Letting y(i) denote the proportion of females of race i, this condition can be written

$$\sum_i y(i)M(i,ij) = x(j) \quad \text{for all j}$$

Assuming the same distribution of men and women (x = y), you can show that the second condition holds by substituting in the matching probabilities specified above. But this condition does not hold more generally. Thus, the model would need to be respecified to allow for different distributions of men and women.